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ω-MORASSES, AND A WEAK FORM OF MARTIN'S AXIOM PROVABLE IN ZFC

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ABSTRACT. We prove, in ZFC, that simplified gap-1 morasses of height ω exist. By earlier work on the relationship between morasses and forcing it immediately follows that a certain Martin's axiom-type forcing axiom is provable in ZFC. We show that this forcing axiom can be thought of as a weak form of MA_{ω_1} and give some applications.

1. Introduction. Gap-1 morasses, as defined by Jensen, are quite complicated, and the smallest height possible for Jensen's morasses is ω_1. In **[4]** we defined a simplified version of gap-1 morasses which has the added advantage that the definition makes sense for morasses of height ω. For regular $\kappa > \omega$, the existence of a $(\kappa, 1)$-morass (a gap-1 morass of height κ) is equivalent to the existence of a simplified $(\kappa, 1)$-morass. Thus, simplified $(\kappa, 1)$-morasses exist in the constructible universe, but their existence is not provable in ZFC. However, the situation for $\kappa = \omega$ is different. In this paper we prove, in ZFC, that simplified $(\omega, 1)$-morasses exist. This was also proved independently by Mark Bickford and Charlie Mills.

In **[4]** we also stated a Martin's axiom-type forcing axiom which is equivalent to the existence of a simplified $(\kappa, 1)$-morass. When $\kappa = \omega$ this forcing axiom is therefore also provable in ZFC. As we will see, the axiom can be thought of as a weak form of MA_{ω_1}.

For the general definition of simplified $(\kappa, 1)$-morasses, and the general statement of the corresponding forcing axiom, we refer the reader to **[4]**. In this paper we will only be concerned with the case $\kappa = \omega$. We will give the definitions for this case below, but first we need some preliminary definitions. Our set-theoretic notation is standard. In particular, in the definitions below an ordinal number is identified with the set of smaller ordinal numbers. Thus, if n is a natural number, then $n = \{0, 1, \ldots, n-1\}$, and ω is the set of natural numbers.

DEFINITION 1.1. For any set X let id_X be the identity function with domain X. For $s < n < \omega$ we define a function $f(s, n)\colon n \to 2n - s$ as follows:

$$f(s,n)(i) = \begin{cases} i & \text{if } i < s, \\ n + i - s & \text{if } i \geqslant s. \end{cases}$$

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DEFINITION 1.2. If $\mathscr{F}$ and $\mathscr{G}$ are sets of functions let $\mathscr{F} \circ \mathscr{G} = \{f \circ g \mid f \in \mathscr{F}, g \in \mathscr{G}\}$.

We are now ready to define the morasses we will be working with. To simplify our terminology, in this paper we will call these structures ω-morasses; in the terminology of [4], they would be called expanded simplified $(\omega, 1)$-morasses.

DEFINITION 1.3. Suppose $\langle n_i \mid i < \omega \rangle$ is an increasing sequence of natural numbers; for each $i < j < \omega$, $\mathscr{F}_{ij}$ is a set of order preserving functions from n_i to n_j; and, for each $i < \omega$, $\mathscr{F}_i$ is a set of order preserving functions from n_i to ω_1. Then $\langle \langle n_i \mid i < \omega \rangle, \langle \mathscr{F}_{ij} \mid i < j < \omega \rangle, \langle \mathscr{F}_i \mid i < \omega \rangle \rangle$ is an *ω-morass* if:

(1) $\forall i < \omega \, \exists s < n_i \, (n_{i+1} = 2n_i - s \text{ and } \mathscr{F}_{i,i+1} = \{\mathrm{id}_{n_i}, f(s, n_i)\})$.

(2) $\forall i < j < k < \omega \, (\mathscr{F}_{ik} = \mathscr{F}_{jk} \circ \mathscr{F}_{ij})$.

(3) $\forall i < j < \omega \, (\mathscr{F}_i = \mathscr{F}_j \circ \mathscr{F}_{ij})$.

(4) $\forall i < \omega \, \forall f_1, f_2 \in \mathscr{F}_i \, \exists j > i \, \exists f_1', f_2' \in \mathscr{F}_{ij} \, \exists g \in \mathscr{F}_j \, (f_1 = g \circ f_1' \text{ and } f_2 = g \circ f_2')$.

(5) $\omega_1 = \bigcup \{ f'' n_0 \mid f \in \mathscr{F}_0 \}$.

Our main theorem on ω-morasses is

THEOREM 1.4. (*ZFC*) *There is an ω-morass.*

Before presenting the weak form of Martin's axiom which follows from Theorem 1.4, we briefly review the basic terminology of forcing and the statement of Martin's axiom. For a more thorough treatment, the reader should see [3]. Suppose $\mathbf{P} = \langle P, \leqslant \rangle$ is a partial order. We will refer to the elements of P as *forcing conditions*, and, if p and q are forcing conditions and $p \leqslant q$, we will say p is an *extension* of q, or *p extends q*. (Usually the elements of P are partial specifications of some mathematical structure, and $p \leqslant q$ means p specifies everything q does, plus perhaps some more.) We call p and q *compatible* in P if they have a common extension—i.e., if some forcing condition extends both of them. Otherwise, they are *incompatible*. An *antichain* is a set of pairwise incompatible conditions. $\mathbf{P}$ has the *countable chain condition* (c.c.c.) if it has no uncountable antichains.

A set $D \subseteq P$ is *dense* in P if every condition has an extension in D. D is *open* if every extension of an element of D is also in D. A set $G \subseteq P$ is *directed* if every pair of elements of G has a common extension which is also in G. If $\mathscr{D}$ is a family of dense subsets of P then G is **P**-*generic over $\mathscr{D}$* if G is directed and G intersects every element of $\mathscr{D}$.

For any cardinal number κ, *Martin's axiom for κ* (MA_κ) is the assertion that if $\mathbf{P}$ is any partial order with the c.c.c., $\mathscr{D}$ is a family of dense sets and $|\mathscr{D}| \leqslant \kappa$, then there is a set G which is $\mathbf{P}$-generic over $\mathscr{D}$. *Martin's axiom* (MA) says that MA_κ holds for every $\kappa < 2^{\aleph_0}$.

Now, suppose $\mathbf{P} = \langle P, \leqslant \rangle$ is a partial order and $\mathscr{D} = \{D_\alpha \mid \alpha < \omega_1\}$ is an indexed family of ω_1 dense open sets. For $p \in P$ let the *realm* of $p = \mathrm{rlm}(p) = \{\alpha < \omega_1 \mid p \in D_\alpha\}$, and for $\alpha < \omega_1$ let $P_\alpha = \{p \in P \mid \mathrm{rlm}(p) \subseteq \alpha\}$. Let $P^* = \bigcup_{n < \omega} P_n$.

DEFINITION 1.5. $\mathscr{D}$ is *weakly ω-indiscernible* if, for each $n < \omega$, $\alpha < \omega_1$ and order preserving function $f: n \to \alpha$, we can choose a function $\sigma_f: P_n \to P_\alpha$ so that the following requirements are satisfied:

(1) If $n < \omega$, $\alpha < \omega_1$, $f: n \to \alpha$ is order preserving and $p \in P_n$, then $\mathrm{rlm}(\sigma_f(p)) = f'' \, \mathrm{rlm}(p)$.

(2) If $n < \omega$, $\alpha < \omega_1$ and $f\colon n \to \alpha$ is order preserving, then σ_f is order preserving; i.e., if $p, q \in P_n$ and $p \leqslant q$, then $\sigma_f(p) \leqslant \sigma_f(q)$.

(3) If $n_1, n_2 < \omega$, $\alpha < \omega_1$ and $f_1\colon n_1 \to n_2$ and $f_2\colon n_2 \to \alpha$ are both order preserving, then $\sigma_{f_2 \circ f_1} = \sigma_{f_2} \circ \sigma_{f_1}$.

(4) $P^* \neq 0$, and $\forall n < \omega$, $D_n \cap P^*$ is dense in P^*.

(5) If $s < n < \omega$, $f = f(s, n)$ and $p \in P_n$, then p and $\sigma_f(p)$ are compatible in P^*.

Our forcing axiom is the statement:

(∗) If **P** is a partial order and $\mathscr{D}$ is weakly ω-indiscernible, then there is a set G that is **P**-generic over $\mathscr{D}$.

In the terminology of [4], (∗) would be called "SFA$_\omega$". ("SFA" stands for "strong forcing axiom". A weak form of the forcing axiom is also considered in [4].)

We will give the proof of Theorem 1.4 in §2. In §3 we discuss the forcing axiom (∗) and give some applications.

2. Proof of Theorem 1.4. Let $\langle n_i | i < \omega\rangle$ and $\langle s_i | i < \omega\rangle$ be sequences of natural numbers such that $\forall i < \omega$ ($s_i < n_i$ and $n_{i+1} = 2n_i - s_i$) and every natural number appears infinitely often in the sequence $\langle s_i | i < \omega\rangle$. For every $i < \omega$ let $\mathscr{F}_{i,i+1} = \{\mathrm{id}_{n_i}, f(s_i, n_i)\}$, and for $j > i + 1$ let $\mathscr{F}_{ij} = \mathscr{F}_{j-1,j} \circ \mathscr{F}_{j-2,j-1} \circ \cdots \circ \mathscr{F}_{i,i+1}$. It is easy to see that 1.3(1) and (2) are satisfied. The hard part of the proof is choosing the sets $\mathscr{F}_{ij}$ so that 1.3(3)–(5) hold. Before we choose these sets we verify some simple properties of what we have constructed so far. Most of these are consequences of 1.3(1) and (2).

LEMMA 2.1. $\forall i < j < \omega\ (\mathrm{id}_{n_i} \in \mathscr{F}_{ij})$.

PROOF. By induction on j. For $j = i + 1$ this follows from 1.3(1). For the induction step, use 1.3(2). □

LEMMA 2.2. $\forall i < j < k < \omega\ (\mathscr{F}_{ij} \subseteq \mathscr{F}_{ik})$.

PROOF. Suppose $f \in \mathscr{F}_{ij}$. By Lemma 2.1, $\mathrm{id}_{n_j} \in \mathscr{F}_{jk}$, so by 1.3(2), $f = \mathrm{id}_{n_j} \circ f \in \mathscr{F}_{jk} \circ \mathscr{F}_{ij} = \mathscr{F}_{ik}$. □

LEMMA 2.3. $\forall i < j < k < m < \omega\ (\mathscr{F}_{km} \circ \mathscr{F}_{ij} \subseteq \mathscr{F}_{im})$.

PROOF. Use Lemma 2.2 and 1.3(2). □

LEMMA 2.4. *Suppose* $i < j < \omega$, $f_1, f_2 \in \mathscr{F}_{ij}$, $a_1, a_2 < n_i$ *and* $f_1(a_1) = f_2(a_2)$. *Then* $a_1 = a_2$ *and* $f_1 \upharpoonright (a_1 + 1) = f_2 \upharpoonright (a_1 + 1)$.

PROOF. By induction on j, using 1.3(1) and (2). For more details, see [4, Lemma 3.2]. □

LEMMA 2.5. $\forall i < j < \omega\ (n_j = \bigcup\{f''n_i | f \in \mathscr{F}_{ij}\})$.

PROOF. By induction on j, using 1.3(1) and (2). □

DEFINITION 2.6. Suppose $a < n < \omega$ and $f\colon n \to \omega_1$. We define a function $f/a\colon n \to \omega_1$ as follows:

$$(f/a) \upharpoonright a = f \upharpoonright a,$$
$$(f/a)(a + i) = f(a) + i \quad \text{for all } i < n - a.$$

LEMMA 2.7. $\forall i < j < \omega \forall f \in \mathscr{F}_{ij} \forall a < n_i\,(f/a \in \mathscr{F}_{ij})$.

PROOF. By induction on j. For $j = i + 1$, the lemma is true because if $a < s_i < n_i$ then $f(s_i, n_i)/a = \mathrm{id}_{n_i}$, if $s_i \leqslant a < n_i$ then $f(s_i, n_i)/a = f(s_i, n_i)$, and, for every $a < n_i$, $\mathrm{id}_{n_i}/a = \mathrm{id}_{n_i}$. For the induction step use 1.3(2) and the fact that, in general, $(f \circ g)/a = (f/g(a)) \circ (g/a)$. □

DEFINITION 2.8. Suppose $k < m < \omega$. We will say that the interval $[k\,, m)$ is *large* if $\forall a < n_k \exists i \in [k\,, m)\,(s_i = a)$.

LEMMA 2.9. $\forall k < \omega \exists m > k$ ($[k\,, m)$ *is large*).

PROOF. This follows from the fact that every number appears infinitely often in the sequence $\langle s_i | i < \omega \rangle$. □

DEFINITION 2.10. Suppose α is an ordinal and, for each $i < \omega$, $\mathscr{F}_i$ is a set of order preserving functions from n_i to α. Then $\langle \mathscr{F}_i | i < \omega \rangle$ is an *α-construction* if 1.3(3)–(5) hold, except with α in place of ω_1 in (5). The α-construction is *good* if, in addition, $\forall i < \omega \forall f \in \mathscr{F}_i \exists m > k > i \exists g \in \mathscr{F}_{ik} \exists h \in \mathscr{F}_m\,(f = h \circ g$ and $[k\,, m)$ is large).

LEMMA 2.11. *Suppose $\langle \mathscr{F}_i | i < \omega \rangle$ is an α-construction, $i < \omega$, f_1, $f_2 \in \mathscr{F}_i$, $a_1, a_2 < n_i$ and $f_1(a_1) = f_2(a_2)$. Then $a_1 = a_2$ and $f_1 \restriction (a_1 + 1) = f_2 \restriction (a_1 + 1)$.*

PROOF. By 1.3(4) choose $j > i$, f_1', $f_2' \in \mathscr{F}_{ij}$, and $g \in \mathscr{F}_j$ such that $f_1 = g \circ f_1'$ and $f_2 = g \circ f_2'$. Now apply Lemma 2.4 to f_1' and f_2'. □

To complete the proof of Theorem 1.4 we must show that there is an ω_1-construction. We do this by building an ω_1-construction inductively in ω_1 steps. For each limit ordinal $\alpha \leqslant \omega_1$ we will choose a good α-construction $\langle \mathscr{F}_i^\alpha | i < \omega \rangle$ such that if $\beta < \alpha$ and β is a limit ordinal then $\forall i < \omega\ (\mathscr{F}_i^\beta \subseteq \mathscr{F}_i^\alpha)$. When we are done, $\langle \langle n_i | i < \omega \rangle, \langle \mathscr{F}_{ij} | i < j < \omega \rangle, \langle \mathscr{F}_i^{\omega_1} | i < \omega \rangle \rangle$ will be an ω-morass.

Since we are only interested in limit ordinals, we start the induction with $\alpha = \omega$. For each $i < \omega$ let $\mathscr{F}_i^\omega = \bigcup_{j > i} \mathscr{F}_{ij}$. Note that by Lemma 2.2 this is the union of an increasing chain and, by Lemma 2.1, $\mathrm{id}_{n_i} \in \mathscr{F}_i^\omega$. Using these two facts, it is easy to verify that 1.3(3) and (4) hold. For (5) (with ω instead of ω_1), use Lemma 2.5 and the fact that $\langle n_i | i < \omega \rangle$ is a strictly increasing sequence. Thus, $\langle \mathscr{F}_i^\omega | i < \omega \rangle$ is an ω-construction. Lemma 2.9 implies that it is good.

If α is a limit of limit ordinals let $\mathscr{F}_i^\alpha = \bigcup \{ \mathscr{F}_i^\beta | \beta < \alpha, \beta$ a limit ordinal$\}$ for each $i < \omega$. It is easy to see that this is a good α-construction extending all previous constructions. Now suppose α is a limit ordinal, $\alpha < \omega_1$, the $\mathscr{F}_i^\alpha$'s have already been chosen and we wish to define $\mathscr{F}_i^{\alpha+\omega}$ for all $i < \omega$. Since $\alpha < \omega_1$ we can fix a sequence $\langle \alpha_i | i < \omega \rangle$ which enumerates α. We will choose an increasing sequence of natural numbers $\langle j_i | i < \omega \rangle$, functions g_i, $g_i' \in \mathscr{F}_{j_i j_{i+1}}$ and $h_i \in \mathscr{F}_{j_i}^\alpha$, and numbers $a_i < n_{j_i}$ such that $\forall i < \omega$

(a) $h_i = h_{i+1} \circ g_i$,
(b) $g_i \restriction a_i = g_i' \restriction a_i$,
(c) $g_i'(a_i) = a_{i+1}$,
(d) $g_i'/a_i = g_i'$,
(e) $g_i'' n_{j_i} \subseteq a_{i+1}$,
(f) $\alpha_i \in h_{i+1}'' n_{j_{i+1}}$,

(g) $\exists k\ (j_i < k < j_{i+1},\ g_i' \in \mathscr{F}_{j_i k}$ and $[k, j_{i+1})$ is large).

Before choosing j_i, g_i, g_i', h_i and a_i, we show how they can be used to complete the proof. So suppose the choice has been made, and (a)–(g) above hold.

For each $i < \omega$ define a function $u_i\colon n_{j_i} \to \alpha + \omega$ as follows:

$$u_i \restriction a_i = h_i \restriction a_i,$$
$$u_i(a_i + b) = \alpha + b \quad \text{for all } b < n_{j_i} - a_i.$$

CLAIM 1. $\forall i < \omega\ (u_i = u_{i+1} \circ g_i')$.

PROOF. First of all, combining the definition of u_i with (a), (b) and (c), we have

$$(u_{i+1} \circ g_i') \restriction a_i = (u_{i+1} \restriction g_i'(a_i)) \circ (g_i' \restriction a_i) = (u_{i+1} \restriction a_{i+1}) \circ (g_i \restriction a_i)$$
$$= (h_{i+1} \circ g_i) \restriction a_i = h_i \restriction a_i = u_i \restriction a_i.$$

Now, suppose $b < n_{j_i} - a_i$. Then by (c) and (d), $g_i'(a_i + b) = a_{i+1} + b$, so

$$(u_{i+1} \circ g_i')(a_i + b) = u_{i+1}(a_{i+1} + b) = \alpha + b = u_i(a_i + b).$$

CLAIM 2. $\forall i < \omega\ (h_i = u_{i+1} \circ g_i)$.

PROOF. Since $g_i'' n_{j_i} \subseteq a_{i+1}$ and $h_{i+1} \restriction a_{i+1} = u_{i+1} \restriction a_{i+1}$, by (a) we have

$$h_i = h_{i+1} \circ g_i = (h_{i+1} \restriction a_{i+1}) \circ g_i = u_{i+1} \circ g_i.$$

Now for $i < \omega$ we define $\mathscr{F}_i^{\alpha+\omega} = \{u_k \circ g \mid j_k > i \text{ and } g \in \mathscr{F}_{i j_k}\}$. In other words, if k_0 is the smallest natural number such that $j_{k_0} > i$, then $\mathscr{F}_i^{\alpha+\omega} = \bigcup_{k \geq k_0}(\{u_k\} \circ \mathscr{F}_{i j_k})$. We prove that this is a good $(\alpha + \omega)$-construction extending our α-construction by verifying a series of claims.

CLAIM 3. $\forall i < \omega\ (u_i \in \mathscr{F}_{j_i}^{\alpha+\omega})$.

PROOF. By Claim 1, $u_i = u_{i+1} \circ g_i'$ and $g_i' \in \mathscr{F}_{j_i j_{i+1}}$.

CLAIM 4. If $f \in \mathscr{F}_i^{\alpha+\omega}$, then $\exists k < \omega\, \forall m \geq k\, \exists g \in \mathscr{F}_{i j_m}\ (f = u_m \circ g)$.

PROOF. Choose k and h such that $i < j_k$, $h \in \mathscr{F}_{i j_k}$ and $f = u_k \circ h$. We show that this k works. Clearly the conclusion holds for $m = k$. For larger m, we use induction. Suppose $m \geq k$ and the conclusion holds for m. Then for some $g \in \mathscr{F}_{i j_m}$, $f = u_m \circ g$. Then by Claim 1, $f = u_m \circ g = u_{m+1} \circ g_m' \circ g$, and $g_m' \circ g \in \mathscr{F}_{j_m j_{m+1}} \circ \mathscr{F}_{i j_m} = \mathscr{F}_{i j_{m+1}}$. Thus the conclusion also holds for $m + 1$.

Note that Claim 4 implies that if k_0 is *any* number such that $j_{k_0} > i$, then $\mathscr{F}_i^{\alpha+\omega} = \bigcup_{k \geq k_0}(\{u_k\} \circ \mathscr{F}_{i j_k})$.

CLAIM 5. $\forall i < \omega\ (\mathscr{F}_i^{\alpha} \subseteq \mathscr{F}_i^{\alpha+\omega})$.

PROOF. Suppose $f \in \mathscr{F}_i^{\alpha}$. By (a) and (f) we can choose $k < \omega$ large enough so that $i < j_k$ and $f(n_i - 1) \in h_k'' n_{j_k}$. By 1.3(3) choose $g \in \mathscr{F}_{i j_k}$ and $h \in \mathscr{F}_{j_k}^{\alpha}$ so that $f = h \circ g$. Let $a_1 = g(n_i - 1)$ and $a_2 = h_k^{-1}(f(n_i - 1))$. Then $h(a_1) = (h \circ g)(n_i - 1) = f(n_i - 1) = h_k(a_2)$, so, by Lemma 2.11, $a_1 = a_2$ and $h \restriction (a_1 + 1) = h_k \restriction (a_1 + 1)$. Since $g'' n_i \subseteq g(n_i - 1) + 1 = a_1 + 1$,

$$f = h \circ g = (h \restriction (a_1 + 1)) \circ g = h_k \circ g.$$

Thus, by Claim 2, $f = h_k \circ g = u_{k+1} \circ g_k \circ g$ and $g_k \circ g \in \mathscr{F}_{j_k j_{k+1}} \circ \mathscr{F}_{i j_k} = \mathscr{F}_{i j_{k+1}}$, so $f \in \mathscr{F}_i^{\alpha+\omega}$.

CLAIM 6. $\forall i < j < \omega\ (\mathscr{F}_i^{\alpha+\omega} = \mathscr{F}_j^{\alpha+\omega} \circ \mathscr{F}_{ij})$.

PROOF. Choose k_0 large enough that $j_{k_0} > j$. By the remark after Claim 4,

$$\mathscr{F}_i^{\alpha+\omega} = \bigcup_{k \geqslant k_0} (\{u_k\} \circ \mathscr{F}_{ij_k}) = \bigcup_{k \geqslant k_0} (\{u_k\} \circ \mathscr{F}_{jj_k} \circ \mathscr{F}_{ij})$$

$$= \left[\bigcup_{k \geqslant k_0} (\{u_k\} \circ \mathscr{F}_{jj_k}) \right] \circ \mathscr{F}_{ij} = \mathscr{F}_j^{\alpha+\omega} \circ \mathscr{F}_{ij}.$$

CLAIM 7. Suppose $i < \omega$ and $f_1, f_2 \in \mathscr{F}_i^{\alpha+\omega}$. Then $\exists j > i \exists f_1', f_2' \in \mathscr{F}_{ij} \exists g \in \mathscr{F}_j^{\alpha+\omega}$ $(f_1 = g \circ f_1'$ and $f_2 = g \circ f_2')$.

PROOF. By Claim 4 we can choose k sufficiently large that $j_k > i$ and for some $f_1', f_2' \in \mathscr{F}_{ij_k}, f_1 = u_k \circ f_1'$ and $f_2 = u_k \circ f_2'$. By Claim 3, $u_k \in \mathscr{F}_{j_k}^{\alpha+\omega}$, so we are done.

CLAIM 8. $\forall i < \omega\, (n_{j_i} - a_i < n_{j_{i+1}} - a_{i+1})$.

PROOF. By (g), choose k such that $j_i < k < j_{i+1}$ and $g_i' \in \mathscr{F}_{j_i k}$. Since g_i' is an order preserving function from n_{j_i} to n_k and $g_i'(a_i) = a_{i+1}$, $n_{j_i} - a_i \leqslant n_k - a_{i+1}$. Since the sequence $\langle n_i | i < \omega \rangle$ is strictly increasing, $n_k < n_{j_{i+1}}$, so $n_{j_i} - a_i < n_{j_{i+1}} - a_{i+1}$.

CLAIM 9. $\bigcup\{ f''n_0 | f \in \mathscr{F}_o^{\alpha+\omega}\} = \alpha + \omega$.

PROOF. Since $\mathscr{F}_0^{\alpha} \subseteq \mathscr{F}_0^{\alpha+\omega}$ we know $\alpha \subseteq \bigcup\{ f''n_0 | f \in \mathscr{F}_0^{\alpha+\omega}\}$. Thus we only need to show $\forall b < \omega \exists f \in \mathscr{F}_0^{\alpha+\omega}\, (\alpha + b \in f''n_0)$. So suppose $b < \omega$. By Claim 8 choose i large enough so that $n_{j_i} - a_i > b$. Then $u_i(a_i + b) = \alpha + b$. By Lemma 2.5 choose $g \in \mathscr{F}_{0j_i}$ such that $a_i + b \in g''n_0$. Then $u_i \circ g \in \mathscr{F}_0^{\alpha+\omega}$ and $\alpha + b \in (u_i \circ g)''n_0$, as required.

Claims 6, 7 and 9 show that $\langle \mathscr{F}_i^{\alpha+\omega} | i < \omega \rangle$ is an $(\alpha + \omega)$-construction, and Claim 5 says that it extends the α-construction $\langle \mathscr{F}_i^{\alpha} | i < \omega \rangle$. The only inductive hypothesis still to be checked is that the construction is good.

CLAIM 10. $\langle \mathscr{F}_i^{\alpha+\omega} | i < \omega \rangle$ is good.

PROOF. Suppose $f \in \mathscr{F}_i^{\alpha+\omega}$. Choose m and g such that $j_m > i$, $g \in \mathscr{F}_{ij_m}$ and $f = u_m \circ g$. By (g) choose k such that $j_m < k < j_{m+1}$, $g_m' \in \mathscr{F}_{j_m k}$ and $[k, j_{m+1})$ is large. Then $f = u_m \circ g = u_{m+1} \circ g_m' \circ g$, $g_m' \circ g \in \mathscr{F}_{ik}$, $u_{m+1} \in \mathscr{F}_{j_{m+1}}^{\alpha+\omega}$ and $[k, j_{m+1})$ is large, as required in the definition of good.

This completes stage $\alpha + \omega$ of the induction, except for the choice of j_i, g_i, g_i', h_i and a_i satisfying (a)–(g). This choice is made by induction on i. To start the induction, let $j_0 = 0$, $h_0 = \mathrm{id}_{n_0}$ and $a_0 = 0$.

Now suppose j_i, h_i and a_i have been chosen. We will show how to choose g_i, g_i', j_{i+1}, h_{i+1} and a_{i+1}. First, by 1.3(3) and (5) we can choose $f \in \mathscr{F}_{j_i}^{\alpha}$ such that $\alpha_i \in f''n_{j_i}$. By 1.3(4) choose $k > j_i$, $h', f' \in \mathscr{F}_{j_i k}$ and $p \in \mathscr{F}_k^{\alpha}$ such that $h_i = p \circ h'$ and $f = p \circ f'$. Now by two applications of the hypothesis that $\langle \mathscr{F}_i^{\alpha} | i < \omega \rangle$ is good we choose $m_2 > k_2 > m_1 > k_1 > k$, $q \in \mathscr{F}_{kk_1}$, $r \in \mathscr{F}_{m_1 k_2}$ and $t \in \mathscr{F}_{m_2}^{\alpha}$ such that $p = t \circ r \circ q$ and the intervals $[k_1, m_1)$ and $[k_2, m_2)$ are both large. Since $[k_1, m_1)$ is large, we can choose $c \in [k_1, m_1)$ such that $s_c = q(h'(a_i))$.

Let $j_{i+1} = m_2$, $h_{i+1} = t$, $g_i = r \circ q \circ h'$, $g_i' = (r \circ f(s_c, n_c) \circ q \circ h')/a_i$ and $a_{i+1} = g_i'(a_i)$. By 1.3(1) and (2) and Lemmas 2.2, 2.3 and 2.7 we have $g_i, g_i' \in \mathscr{F}_{j_i k_2} \subseteq \mathscr{F}_{j_i j_{i+1}}$. We verify (a)–(g) one at a time.

(a) $h_i = p \circ h' = t \circ r \circ q \circ h' = h_{i+1} \circ g_i$.

(b) By the definition of $f(s_c, n_c)$ and the choice of c,

$$f(s_c, n_c) \upharpoonright q(h'(a_i)) = f(s_c, n_c) \upharpoonright s_c = \mathrm{id}_{s_c}.$$

Therefore

$$g_i' \restriction a_i = (r \circ f(s_c, n_c) \circ q \circ h') \restriction a_i = (r \circ q \circ h') \restriction a_i = g_i \restriction a_i.$$

(c) Clear.

(d) Clear.

(e) Since $q \circ h' \in \mathscr{F}_{j_i k_1}$,

$$(q \circ h')''n_{j_i} \subseteq n_{k_1} \leqslant n_c = f(s_c, n_c)(s_c) = f(s_c, n_c)(q(h'(a_i))).$$

Therefore

$$g_i''n_{j_i} = (r \circ q \circ h')''n_{j_i} \subseteq r(f(s_c, n_c)(q(h'(a_i)))) = g_i'(a_i) = a_{i+1}.$$

(f) Since $f = p \circ f' = t \circ r \circ q \circ f' = h_{i+1} \circ r \circ q \circ f'$, $\alpha_i \in f''n_{j_i} \subseteq h_{i+1}''n_{j_{i+1}}$.

(g) $g_i' \in \mathscr{F}_{j_i k_2}$ and $[k_2, j_{i+1}) = [k_2, m_2)$ is large.

This completes the proof of Theorem 1.4. □

3. Weak MA and applications. By Theorem 1.4, we can fix, for the rest of the paper, an ω-morass $\langle\langle n_i | i < \omega\rangle, \langle \mathscr{F}_{ij} | i < j < \omega\rangle, \langle \mathscr{F}_i | i < \omega\rangle\rangle$. Before showing how we can use this ω-morass to prove the forcing axiom (∗) stated in §1, let us first see why (∗) is a weak form of MA_{ω_1}. We will need the following combinatorial lemma, a proof of which can be found in [**3**].

LEMMA 3.1 (Δ-SYSTEM LEMMA). *Suppose A is an uncountable set of finite subsets of ω_1. Then there is an uncountable $B \subseteq A$ and a set $r \subseteq \omega_1$ such that r is an initial segment of every element of B, and if $x, y \in B$ and $x \neq y$, then either $x \subseteq \min(y \setminus r)$ or $y \subseteq \min(x \setminus r)$.*

THEOREM 3.2. *Suppose $\mathbf{P} = \langle P, \leqslant \rangle$ is a partial order and $\mathscr{D} = \{D_\alpha | \alpha < \omega_1\}$ is weakly ω-indiscernible. Then there is a c.c.c. suborder $\mathbf{Q} = \langle Q, \leqslant\rangle$ of $\mathbf{P}$ such that $\forall \alpha < \omega_1$ ($Q \cap D_\alpha$ is dense in $\mathbf{Q}$).*

PROOF. Let P_α, P^* and σ_f be as in §1. First we choose a nonempty countable set $A \subseteq P^*$ such that:

(1) If $n < \omega$, f: $n \to \omega$ is order preserving and $p \in A \cap P_n$, then $\sigma_f(p) \in A$.

(2) If $p \in A$ and $n < \omega$, then $\exists q \in A \cap D_n$ ($q \leqslant p$).

(3) If $s < n < \omega$ and $p \in A \cap P_n$, then $\exists q \in A$ ($q \leqslant p$ and $q \leqslant \sigma_{f(s,n)}(p)$).

It is not hard to build up such a set in countably many steps, using (1), (4) and (5) in the definition of weak ω-indiscernibility. Now let $Q = \{\sigma_f(p) | \exists n < \omega$ ($p \in A \cap P_n$ and f: $n \to \omega_1$ is order preserving)$\}$.

To see that $\mathbf{Q} = \langle Q, \leqslant \rangle$ is c.c.c., suppose $X \subseteq Q$ and $|X| = \omega_1$. Since A is countable, we can assume that for some fixed $n < \omega$ and $p \in A \cap P_n$, $X = \{\sigma_{f_\alpha}(p) | \alpha < \omega_1\}$, where for each $\alpha < \omega_1$, f_α: $n \to \omega_1$ is order preserving. Now apply the Δ-system lemma to the set of ranges of the f_α's to choose $\alpha, \beta < \omega_1$ and $s < n$ such that $f_\alpha \restriction s = f_\beta \restriction s$ and $f_\alpha''n \subseteq f_\beta(s)$.

Let $R = f_\alpha''n \cup f_\beta''n$ and $m = |R| = 2n - s$, and let g: $m \to R$ enumerate R, in order. It is not hard to see that $f_\alpha = g \circ \mathrm{id}_n = g \restriction n$ and $f_\beta = g \circ f(s, n)$. Let $q = \sigma_{\mathrm{id}_n}(p) \in A \cap P_n$ by (1) above, and by (3) above choose $r \in A$ such that $r \leqslant q$ and $r \leqslant \sigma_{f(s,n)}(q)$. Choose $k \geqslant m$ such that $r \in P_k$, and let h: $k \to \omega_1$ be any order preserving function extending g. Then $h \circ \mathrm{id}_n = g \circ \mathrm{id}_n = f_\alpha$ and $h \circ f(s,n) \circ \mathrm{id}_n = g \circ f(s,n) = f_\beta$, so

$$\sigma_h(r) \leqslant \sigma_h(q) = \sigma_{h \circ \mathrm{id}_n}(p) = \sigma_{f_\alpha}(p)$$

and

$$\sigma_h(r) \leqslant \sigma_h(\sigma_{f(s,n)}(q)) = \sigma_{h \circ f(s,n) \circ \mathrm{id}_n}(p) = \sigma_{f_\beta}(p).$$

Therefore $\sigma_{f_\alpha}(p)$ and $\sigma_{f_\beta}(p)$ are compatible in **Q**, as required.

Now fix $\alpha < \omega_1$. To see that $Q \cap D_\alpha$ is dense in **Q**, suppose $n < \omega$, $p \in A \cap P_n$ and $f\colon n \to \omega_1$ is order preserving, so $\sigma_f(p) \in Q$. Let $R = f''n \cup \{\alpha\}$ and $m = |R|$, and let $g\colon m \to R$ enumerate R, in order. Let $q = \sigma_{g^{-1} \circ f}(p) \in A$, by (1) above, and by (2) above choose $r \in A$ such that $r \leqslant q$ and $r \in D_{g^{-1}(\alpha)}$. Choose $k \geqslant m$ so that $r \in P_k$, and let $h\colon k \to \omega_1$ be any order preserving function extending g. Then $\sigma_h(r) \in Q \cap D_{h(g^{-1}(\alpha))} = Q \cap D_\alpha$ and $\sigma_h(r) \leqslant \sigma_h(q) = \sigma_{h \circ g^{-1} \circ f}(p) = \sigma_f(p)$, as required. □

COROLLARY 3.3. $MA_{\omega_1} \to (*)$.

PROOF. Apply MA_{ω_1} to the c.c.c. order **Q** from Theorem 3.2. □

Of course, Corollary 3.3. is superseded by

THEOREM 3.4. (ZFC) $(*)$ *is true*.

PROOF. Since this is really just the case $\kappa = \omega$ of Theorem 4.1 of [**4**], we just give a brief sketch of the proof.

Suppose $\mathbf{P} = \langle P, \leqslant \rangle$ is a partial order, and $\mathscr{D} = \{D_\alpha \mid \alpha < \omega_1\}$ is weakly ω-indiscernible. Let P_α, P^* and σ_f be as in the definition of weak ω-indiscernibility. We will need to modify the functions in the ω-morass slightly, so we introduce the following notation: Suppose $n < \omega$, $f\colon n \to \omega_1$ is order preserving and $\alpha \leqslant \beta$. Then $f_{\alpha\beta}$ is the function with domain $\alpha \cdot n$ defined as follows:

$$f_{\alpha\beta}(\alpha \cdot i + \delta) = \beta \cdot f(i) + \delta \quad \text{for all } i < n, \delta < \alpha.$$

It is easy to see that $f_{\alpha\beta}$ is order preserving.

In order to construct a generic set we choose an increasing sequence of natural numbers $\langle a_i \mid i < \omega \rangle$ and a sequence of forcing conditions $\langle p_i \mid i < \omega \rangle$ such that:

(1) $\forall i < \omega\, (\mathrm{rlm}(p_i) \subseteq a_i \cdot n_i)$.

(2) $\forall i < j < \omega\, \forall f \in \mathscr{F}_{ij}\, (f_{a_i a_j}{}''(a_i \cdot n_i) \subseteq \mathrm{rlm}(p_j))$.

(3) $\forall i < j < \omega\, \forall f \in \mathscr{F}_{ij}\, (p_j \leqslant \sigma_{f_{a_i a_j}}(p_i))$.

We leave the details of the construction to the reader.

Now let $G = \{\sigma_{f_{a_i \omega}}(p_i) \mid i < \omega, f \in \mathscr{F}_i\}$. Using (3) above and 1.3(4) the reader can check that G is directed. To see that G meets all the dense sets use (2) above and 1.3(5). □

To illustrate the use of ω-morasses and $(*)$ we give two examples of applications. Both of the theorems were already known, but the proofs given here are new.

THEOREM 3.5 (KEISLER AND GALVIN, INDEPENDENTLY; SEE [**2**]). *There are tree orders* $<_i$ *on* ω_1 *for* $i < \omega$ *such that*:

(1) $\forall i < \omega\, ((\omega_1, <_i)$ *is a tree of finite height*).

(2) $\forall\alpha, \beta < \omega_1 \forall i < \omega\, (\alpha <_i \beta \to \alpha <_{i+1} \beta \to \alpha < \beta)$.
(3) $\forall\alpha < \beta < \omega_1 \exists i < \omega\, (\alpha <_i \beta)$.

PROOF. One way to prove this is to apply (*) to the obvious partial order for constructing such a sequence of tree orders with finite forcing conditions. However, it is easier to define the tree orders directly from the ω-morass.

For each $i < \omega$ define $<_i$ by

$$\alpha <_i \beta \leftrightarrow \exists f \in \mathscr{F}_i \exists a < b < n_i\, (\alpha = f(a) \text{ and } \beta = f(b)).$$

To prove that $<_i$ is a tree order, we will need the following

CLAIM. Suppose $f \in \mathscr{F}_i$, $b < n_i$ and $\beta = f(b)$. Then $\{\alpha | \alpha <_i \beta\} = f''b$.

PROOF. Clearly if $a < b$, then $f(a) <_i \beta$. Now suppose $\alpha <_i \beta$. Choose $g \in \mathscr{F}_i$ and $a < c < n_i$ such that $g(a) = \alpha$ and $g(c) = \beta$. Then since $g(c) = \beta = f(b)$, by Lemma 2.11, $c = b$ and $g \restriction b = f \restriction b$. Therefore $\alpha = g(a) = f(a) \in f''b$.

To see that $<_i$ is transitive, suppose $\alpha <_i \beta <_i \gamma$. Choose $f \in \mathscr{F}_i$ and $b < c < n_i$ such that $f(b) = \beta$ and $f(c) = \gamma$. Since $\alpha <_i \beta$, by the claim there is some $a < b$ such that $f(a) = \alpha$. Clearly $a < c$, so $\alpha <_i \gamma$. Since all functions in $\mathscr{F}_i$ are order preserving it is clear that if $\alpha <_i \beta$, then $\alpha < \beta$, so $<_i$ is irreflexive.

Now suppose $\beta < \omega_1$, and choose $f \in \mathscr{F}_i$ and $b < n_i$ such that $\beta = f(b)$. By the claim the set of $<_i$-predecessors of β is $f''b$, which is clearly linearly ordered by $<_i$ with order type $b < n_i$. Thus $(\omega_1, <_i)$ is a tree with height n_i.

To verify (2) above, suppose $\alpha <_i \beta$. Choose $f \in \mathscr{F}_i$ and $a < b < n_i$ such that $f(a) = \alpha$ and $f(b) = \beta$. By 1.3(3) we can choose $g \in \mathscr{F}_{i,i+1}$ and $h \in \mathscr{F}_{i+1}$ such that $f = h \circ g$. Then $g(a) < g(b) < n_{i+1}$, $h(g(a)) = f(a) = \alpha$ and $h(g(b)) = f(b) = \beta$, so $\alpha <_{i+1} \beta$. We have already observed that $\alpha <_i \beta \to \alpha < \beta$, so (2) holds.

Finally, for (3) suppose $\alpha < \beta < \omega_1$. Choose $f_1, f_2 \in \mathscr{F}_0$ such that $\alpha \in f_1''n_0$ and $\beta \in f_2''n_0$. By 1.3(4) choose $i > 0$, $f_1', f_2' \in \mathscr{F}_{0i}$ and $g \in \mathscr{F}_i$ such that $f_1 = g \circ f_1'$ and $f_2 = g \circ f_2'$. Then $\alpha, \beta \in g''n_i$, so we can choose $a, b < n_i$ such that $\alpha = g(a)$ and $\beta = g(b)$. Since g is order preserving and $\alpha < \beta$, we must have $a < b$. Therefore $\alpha <_i \beta$. □

For our second application, we need the following

DEFINITION 3.6. If $A, B \subseteq \omega$ then $A \subseteq^* B$ if $A \setminus B$ is finite. An ω_1, ω_1^*-*gap* in $\mathscr{P}(\omega)$/finite is a pair of sequences of subsets of ω, $\langle \langle A_\alpha^0 | \alpha < \omega_1 \rangle, \langle A_\alpha^1 | \alpha < \omega_1 \rangle \rangle$, such that:

(1) $\forall\alpha < \beta < \omega_1\, (A_\alpha^0 \subseteq^* A_\beta^0 \subseteq^* A_\beta^1 \subseteq^* A_\alpha^1)$.
(2) $\neg\exists X \subseteq \omega \forall\alpha < \omega_1\, (A_\alpha^0 \subseteq^* X \subseteq^* A_\alpha^1)$.

THEOREM 3.7 (HAUSDORFF; SEE [1]). *There is an ω_1, ω_1^*-gap in $\mathscr{P}(\omega)$/finite.*

PROOF. Hausdorff constructed his gap from countable pieces. We will apply (*) to a partial order for building a gap from finite pieces.

We will construct characteristic functions for the sets in the ω_1, ω_1^*-gap, rather than constructing the sets themselves. Thus our goal is to construct a function $F: \omega_1 \times 2 \times \omega \to 2$ such that, letting $A_\alpha^i = \{n < \omega | F(\alpha, i, n) = 1\}$ for each $i < 2$ and $\alpha < \omega_1$, $\langle \langle A_\alpha^0 | \alpha < \omega_1 \rangle, \langle A_\alpha^1 | \alpha < \omega_1 \rangle \rangle$ is a gap. Translating 3.6(1) and (2) into properties of the function F, we have the following requirements which must be met:

(1′) $\forall\alpha < \beta < \omega_1 \exists i < \omega \forall j \geqslant i\ (F(\alpha, 0, j) \leqslant F(\beta, 0, j) \leqslant F(\beta, 1, j) \leqslant F(\alpha, 1, j))$.

(2′) $\neg\exists f \in {}^{\omega}2\, \forall\alpha < \omega_1\, \exists i < \omega\, \forall j \geqslant i\, (F(\alpha, 0, j) \leqslant f(j) \leqslant F(\alpha, 1, j))$.

Let us call a function F: $\omega_1 \times 2 \times \omega \to 2$ satisfying (1′) an ω_1, ω_1^*-*sequence*. F is *good* if, in addition,

$$\forall\alpha < \beta < \omega_1\, \exists k < \omega\, (F(\alpha,0,k) = F(\alpha,1,k) = 0 \text{ and } F(\beta,0,k) = 1).$$

LEMMA 3.8. *If F is a good ω_1, ω_1^*-sequence, then* (2′) *holds.*

PROOF. Suppose not. Let f: $\omega \to 2$ be a counterexample, and for each $\alpha < \omega_1$ choose $i_\alpha < \omega$ such that $\forall j \geqslant i_\alpha\, (F(\alpha, 0, j) \leqslant f(j) \leqslant F(\alpha, 1, j))$. Choose $i < \omega$ and an uncountable set $X \subseteq \omega_1$ such that $\forall\alpha \in X\, (i_\alpha = i)$. Now choose $\alpha, \beta \in X$, $\alpha < \beta$, such that $\forall j < i\, (F(\alpha, 0, j) = F(\beta, 0, j))$.

Since F is good, we can choose $k < \omega$ such that $F(\alpha, 0, k) = F(\alpha, 1, k) = 0$ and $F(\beta, 0, k) = 1$. Since $F(\alpha, 0, k) \neq F(\beta, 0, k)$, by the choice of α and β, $k \geqslant i = i_\alpha = i_\beta$. But then $1 = F(\beta, 0, k) \leqslant f(k) \leqslant F(\alpha, 1, k) = 0$, which is a contradiction. □

To complete the proof of Theorem 3.7 we now define a partial order for constructing a good ω_1, ω_1^*-sequence and apply (∗). A forcing condition will be a function p: $X \times 2 \times m \to 2$ for some finite $X \subseteq \omega_1$ and $m < \omega$ which is good "so far"; i.e., we require that if $\alpha, \beta \in X$ and $\alpha < \beta$ then $\exists k < m\ (p(\alpha, 0, k) = p(\alpha, 1, k) = 0$ and $p(\beta, 0, k) = 1)$. Let P be the set of all such conditions, and order P as follows: Suppose $p, q \in P$, where p: $X \times 2 \times m \to 2$ and q: $Y \times 2 \times n \to 2$. Let $p \leqslant q$ iff $q \subseteq p$ and $\forall\alpha, \beta \in Y\, \forall j \in [n, m)\ (\alpha < \beta \to p(\alpha, 0, j) \leqslant p(\beta, 0, j) \leqslant p(\beta, 1, j) \leqslant p(\alpha, 1, j))$. For each $\alpha < \omega_1$ define $D_\alpha \subseteq P$ as follows: If $p \in P$ and p: $X \times 2 \times m \to 2$, then $p \in D_\alpha \leftrightarrow \alpha \in X$. It is not hard to see that $\mathbf{P} = \langle P, \leqslant \rangle$ is a partial order and for every $\alpha < \omega_1$, D_α is dense and open.

Let $\mathrm{rlm}(p)$, P_α and P^* be defined as in §1. Note that if $p \in P$ and p: $X \times 2 \times m \to 2$ then $\mathrm{rlm}(p) = X$. Now suppose $n < \omega$, $\alpha < \omega_1$ and f: $n \to \alpha$ is order preserving. Define σ_f: $P_n \to P_\alpha$ as follows: Suppose $p \in P_n$ and p: $X \times 2 \times m \to 2$. Then $X \subseteq n$. Let $\sigma_f(p) = q$, where q: $(f''X) \times 2 \times m \to 2$ and $\forall a \in X\, \forall i < 2\, \forall j < m\, (q(f(a), i, j) = p(a, i, j))$. Most of the verification of weak ω-indiscernibility is trivial. We will only check 1.5(5).

Suppose $s < n < \omega$ and $p \in P_n$, where p: $X \times 2 \times m \to 2$. Let $f = f(s, n)$ and $q = \sigma_f(p)$. Note that $X \subseteq n$ and q: $(f''X) \times 2 \times m \to 2$. We must define $r \in P^*$ such that $r \leqslant p$ and $r \leqslant q$. We could define $r = p \cup q$, but this might not be good "so far"; i.e., there might be some $a \in X \setminus (f''X)$ and $b \in (f''X) \setminus X$ such that $\neg\exists k < m\, (r(a, 0, k) = r(a, 1, k) = 0$ and $r(b, 0, k) = 1)$. To fix this, we define r: $(X \cup f''X) \times 2 \times (m + 1) \to 2$ so that $p \cup q \subseteq r$ and

$$\begin{aligned}
&r(a,0,m) = 0 && \text{for all } a \in X,\\
&r(a,1,m) = 0 && \text{for all } a \in X\setminus(f''X),\\
&r(a,1,m) = 1 && \text{for all } a \in X \cap (f''X),\\
&r(a,0,m) = r(a,1,m) = 1 && \text{for all } a \in (f''X)\setminus X.
\end{aligned}$$

We let the reader verify that $r \in P^*$ and $r \leqslant p, q$, as required in 1.5(5).

By (∗), there is a set G that is $\mathbf{P}$-generic over $\mathscr{D} = \{D_\alpha \mid \alpha < \omega_1\}$. Let $F = \bigcup G$. It is easy to see that F is a good ω_1, ω_1^*-sequence, so by Lemma 3.8 we are done. □

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